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Note

The competition numbers of complete tripartite graphs

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ABSTRACT

For a graph G, it is known to be a hard problem to compute the competition number $k(G)$ of the graph G in general. In this paper, we give an explicit formula for the competition numbers of complete tripartite graphs.

1. Introduction and main result

Cohen [1] introduced the notion of a competition graph in connection with a problem in ecology in 1968 (also see [2]). The *competition graph* $C(D)$ of a digraph $D = (V, A)$ is an undirected graph $G = (V, E)$ which has the same vertex set V and has an edge between distinct two vertices $x, y \in V$ if there exists a vertex $a \in V$ such that $(x, a), (y, a) \in A$.

Roberts [5] observed that, for any graph, the graph with sufficiently many isolated vertices is the competition graph of some acyclic digraph. The minimum number of such isolated vertices was called the *competition number* of the graph G and was denoted by $k(G)$. It is difficult to compute the competition number of a graph in general as Opsut [4] has shown that the computation of the competition number of a graph is an NP-hard problem.

But, for a graph in some special classes, it is easy to obtain the competition number of the graph. The following are some of known results for competition numbers.

- If G is a chordal graph which has no isolated vertices, then $k(G) = 1$.
- If G is a triangle-free connected graph, then $k(G) = |E(G)| - |V(G)| + 2$.

As corollaries of these results, we have

- $k(K_n) = 1$, $k(K_{n,n}) = n^2 - 2n + 2$, $k(K_{n_1,n_2}) = (n_1 - 1)(n_2 - 1) + 1$.

Competition graphs and the competition numbers of graphs are closely related to edge clique covers and the edge clique cover numbers of the graphs. A *clique* of a graph G is an empty set or a subset of $V(G)$ such that its induced subgraph of G is a complete graph. A clique consisting of three vertices is called a *triangle*. An *edge clique cover* (or an *ECC* for short) of a graph G is a family of cliques of G such that each edge of G is contained in some clique in the family. The minimum size of an edge clique cover of G is called the *edge clique cover number* (or the *ECC number* for short) of the graph G, and is denoted by $\theta_e(G)$.

Opsut [4] showed that, for any graph G, the competition number satisfies an inequality $\theta_e(G) - |V(G)| + 2 \le k(G) \le \theta_e(G)$. Dutton and Brigham [3] showed that a graph G is a competition graph of some digraph if and only if $\theta_e(G) \le |V(G)|$, and also characterized the competition graphs of acyclic digraphs by using ECCs as follows.

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Theorem A. *A graph G is the competition graph of an acyclic digraph if and only if there exist an ordering $v_1, \ldots, v_n$ of the vertices of G and an edge clique cover $\{S_1, \ldots, S_n\}$ of G such that $v_i \in S_j \Rightarrow i < j$.*

For other applications of ECCs, see [6].

In this paper, we give an explicit formula for the competition numbers $k(K_{n,n,n})$ of complete tripartite graphs $K_{n,n,n}$. The following is our main result which will be proven in the following section:

Theorem 1. *For $n \geq 2$, the competition number of the complete tripartite graph $K_{n,n,n}$ is given by the following:*

$$k(K_{n,n,n}) = n^2 - 3n + 4. \tag{1.1}$$

2. Proof of Theorem 1

Let $K_{n,n,n}$ $(n \geq 2)$ be a complete tripartite graph on three disjoint sets $A := \{a_1, \ldots, a_n\}$, $B := \{b_1, \ldots, b_n\}$, and $C := \{c_1, \ldots, c_n\}$.

Put $\triangle(i, j, l) := \{a_i, b_j, c_l\}$ for $1 \leq i, j, l \leq n$. Then $\triangle(i, j, l)$ are triangles of $K_{n,n,n}$. Note that there are n^3 triangles. Let

$$\mathcal{F} := \{\triangle(i, j, l) \mid l = i + j - 1,\ 1 \leq i, j \leq n\}, \tag{2.1}$$

where $i + j - 1$ are reduced modulo n. Note that $|\mathcal{F}| = n^2$.

Lemma 2. *The family $\mathcal{F}$ defined in (2.1) is an edge clique cover of $K_{n,n,n}$ of minimum size. In particular, $\theta_e(K_{n,n,n}) = n^2$.*

Proof. Take any edge a_ib_j between A and B, then both a_i and b_j are in $\triangle(i, j, l) \in \mathcal{F}$, where $l = i + j - 1 \pmod n$. Take any edge a_ic_l between A and C, then both a_i and c_l are in $\triangle(i, j, l) \in \mathcal{F}$, where $j = l - i + 1 \pmod n$. Take any edge b_jc_l between B and C, then both b_j and c_l are in $\triangle(i, j, l) \in \mathcal{F}$, where $i = l - j + 1 \pmod n$. Thus the family $\mathcal{F}$ is an ECC of $K_{n,n,n}$.

Since all maximal cliques of $K_{n,n,n}$ have size 3, we may assume that an ECC of $K_{n,n,n}$ of minimum size consists of triangles. Since $|E(K_{n,n,n})| = 3n^2$ and a triangle has 3 edges, any ECC of $K_{n,n,n}$ has size at least n^2, i.e. $\theta_e(K_{n,n,n}) \geq n^2$. Since $|\mathcal{F}| = n^2$, we conclude $\theta_e(K_{n,n,n}) = n^2$ and thus we have that $\mathcal{F}$ is an ECC of $K_{n,n,n}$ of minimum size. □

Lemma 3. *Let $\mathcal{E}$ be an edge clique cover of $K_{n,n,n}$ of minimum size. If two cliques $S, S' \in \mathcal{E}$ are distinct, then we have $|S \cap S'| \leq 1$.*

Proof. Suppose that there exist cliques $S, S' \in \mathcal{E}$ such that $|S \cap S'| \geq 2$ and $S \neq S'$. Then we have $|S \cap S'| = 2$ since any maximal clique of $K_{n,n,n}$ has size 3. Here we know that $|\mathcal{E}| = n^2$ by Lemma 2. Now we count the number of edges which are covered by $\mathcal{E}$. The two cliques S and S' cover at most 5 edges since $|S \cap S'| = 2$, and $\mathcal{E} \setminus \{S, S'\}$ covers at most $3(n^2 - 2)$ edges. Thus the family $\mathcal{E}$ covers at most $5 + 3(n^2 - 2) = 3n^2 - 1$ edges of $K_{n,n,n}$. On the other hand, we know that $|E(K_{n,n,n})| = 3n^2$. This is a contradiction to the hypothesis that $\mathcal{E}$ is an ECC of $K_{n,n,n}$. Hence the lemma holds. □

Lemma 4. *We can label the vertices of $K_{n,n,n}$ as $v_1, \ldots, v_{3n}$, and choose $3n - 3$ triangles $\triangle_1, \ldots, \triangle_{3n-3}$ from the family $\mathcal{F}$ defined in (2.1) so that*

$$\triangle_1 \cup \cdots \cup \triangle_i \subseteq \{v_1, \ldots, v_{i+3}\} \tag{2.2}$$

for $1 \leq i \leq 3n - 3$.

Proof. We label the vertices of $K_{n,n,n}$ as $v_1, v_2, \ldots, v_{3n}$ in the following order:

$$a_1, b_1, c_1, a_2, b_n, c_n, a_n, b_2, c_2, a_{n-1}, b_{n-1}, c_{n-1}, a_{n-2}, b_{n-2}, c_{n-2}, \ldots, a_3, b_3, c_3. \tag{2.3}$$

More precisely, we put $v_1, \ldots, v_9$ as above, and $v_{3s+7} = a_{n-s}$, $v_{3s+8} = b_{n-s}$, $v_{3s+9} = c_{n-s}$ for $1 \leq s \leq n - 3$. Now choose triangles from $\mathcal{F}$ and label them as follows:

$$\begin{array}{lll}
\triangle_1 = \{a_1, b_1, c_1\}, & \triangle_2 = \{a_2, b_n, c_1\}, & \triangle_3 = \{a_1, b_n, c_n\}, \\
\triangle_4 = \{a_n, b_1, c_n\}, & \triangle_5 = \{a_n, b_2, c_1\}, & \triangle_6 = \{a_1, b_2, c_2\}, \\
\triangle_7 = \{a_{n-1}, b_2, c_n\}, & \triangle_8 = \{a_2, b_{n-1}, c_n\}, & \triangle_9 = \{a_1, b_{n-1}, c_{n-1}\}, \\
\ldots, & \ldots, & \ldots, \\
\triangle_{3s+4} = \{a_{n-s}, b_2, c_{n-s+1}\}, & \triangle_{3s+5} = \{a_2, b_{n-s}, c_{n-s+1}\}, & \triangle_{3s+6} = \{a_1, b_{n-s}, c_{n-s}\}, \\
\qquad = \{v_{3s+7}, v_6, v_{3s+6}\}, & \qquad = \{v_4, v_{3s+8}, v_{3s+6}\}, & \qquad = \{v_1, v_{3s+8}, v_{3s+9}\}, \\
\ldots, & \ldots, & \ldots, \\
\triangle_{3n-5} = \{a_3, b_2, c_4\}, & \triangle_{3n-4} = \{a_2, b_3, c_4\}, & \triangle_{3n-3} = \{a_1, b_3, c_3\},
\end{array}$$

where $1 \leq s \leq n - 3$. Note that $\triangle_i$ are all distinct. Now, we will show that (2.2) holds. For $i = 1, \ldots, 6$, we can easily check that (2.2) holds. For $i = 7, \ldots, 3n - 3$, it can easily be seen that the vertex of maximum index in $\triangle_i$ has index at most $i + 3$. Thus, we conclude that $\triangle_1 \cup \cdots \cup \triangle_i \subseteq \{v_1, \ldots, v_{i+3}\}$ for $1 \leq i \leq 3n - 3$. Hence the lemma holds. □

Now we are ready to prove our main theorem.

Proof of Theorem 1. First, we will show $k(K_{n,n,n}) \geq n^2 - 3n + 4$. Let $k = k(K_{n,n,n})$ for convenience. Then the graph $G := K_{n,n,n} \cup I_k$ is the competition graph of some acyclic digraph D, where I_k denotes a set of k isolated vertices. Then, by Theorem A, we can label the vertices of G as $v_1, \ldots, v_{3n+k}$ so that there exists an ECC $\mathcal{S} = \{S_1, \ldots, S_{3n+k}\}$ of G satisfying $v_i \in S_j \Rightarrow i < j$. That is, $S_j \subseteq \{v_1, \ldots, v_{j-1}\}$. Since I_k is the set of isolated vertices of G, $\mathcal{S}$ is an ECC of $G - I_k$. Since $K_{n,n,n} = G - I_k$, $\mathcal{S}$ is an ECC of $K_{n,n,n}$. Since any edge of G is contained in a triangle and any maximal clique of G has size 1 or 3, we may assume that any nonempty clique S_i is a triangle. Therefore we may assume that $S_1 = S_2 = S_3 = \emptyset$. Thus $\mathcal{T} := \mathcal{S} \setminus \{S_1, S_2, S_3\}$ is an ECC of $K_{n,n,n}$. Since $S_4 \subseteq \{v_1, v_2, v_3\}$ and $S_5 \subseteq \{v_1, v_2, v_3, v_4\}$, one of the following is true:

(i) S_4 or S_5 is empty;
(ii) $S_4 = S_5$;
(iii) $|S_4 \cap S_5| = 2$.

If $|S_4 \cap S_5| = 2$, then $\mathcal{T}$ is an ECC of $K_{n,n,n}$ that is not of the minimum size by Lemma 3. If (i) or (ii) holds, then it is obvious that $\mathcal{T}$ is an ECC of $K_{n,n,n}$ that is not of the minimum size. Thus $3n + k - 3 > n^2$ by Lemma 2. Since k and n are integers, it holds that $k \geq n^2 - 3n + 4$.

Now we show that $k(K_{n,n,n}) \leq n^2 - 3n + 4$. By Lemma 4, there exist a labeling $v_1, \ldots, v_{3n}$ of the vertices of $K_{n,n,n}$, and triangles $\triangle_1, \ldots, \triangle_{3n-3} \in \mathcal{F}$ such that $\triangle_1 \cup \cdots \cup \triangle_i \subseteq \{v_1, \ldots, v_{i+3}\}$ for $1 \leq i \leq 3n - 3$. Since $|\mathcal{F}| = n^2$, there are $n^2 - 3n + 3$ triangles in $\mathcal{F} \setminus \{\triangle_1, \ldots, \triangle_{3n-3}\}$. Label those triangles as $T_1, T_2, \ldots, T_{n^2-3n+3}$. Now, we define a digraph D as follows:

$$V(D) = \{v_1, \ldots, v_{3n}\} \cup \{z_0, z_1, \ldots, z_{n^2-3n+3}\},$$

$$A(D) = \bigcup_{i=1}^{3n-4} \{(x, v_{i+4}) \mid x \in \triangle_i\} \cup \{(x, z_0) \mid x \in \triangle_{3n-3}\} \cup \bigcup_{i=1}^{n^2-3n+3} \{(x, z_i) \mid x \in T_i\}.$$

Then this digraph D is acyclic. For, vertex z_i has no outgoing arcs for each $i = 0, \ldots, n^2 - 3n + 3$ and $(v_i, v_j) \in A(D) \Rightarrow i < j$. Since every clique in the ECC $\mathcal{F}$ has a common out-neighbor in D, we have $E(K_{n,n,n}) \subseteq E(C(D))$. On the other hand, the in-neighborhood of each vertex in D is either empty or a clique in $\mathcal{F}$, it is true that $E(K_{n,n,n}) \supseteq E(C(D))$. Thus $C(D) = K_{n,n,n} \cup \{z_0, z_1, \ldots, z_{n^2-3n+3}\}$. Hence we have $k(K_{n,n,n}) \leq n^2 - 3n + 4$.

Therefore, $k(K_{n,n,n}) = n^2 - 3n + 4$ holds. $\square$

3. Concluding remarks

In this paper, we compute the competition numbers of complete tripartite graphs on the vertex sets of the same size. We present the following problems for further study:

- What is the competition number of a complete tripartite graphs K_{n_1,n_2,n_3} on the vertex sets of different size?
- What is the competition number of the complete tetrapartite graphs $K_{n,n,n,n}$ (on the vertex sets of the same size)?
- More generally, what is the competition number of a complete multipartite graph $K_{n_1,n_2,\ldots,n_m}$?

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